ISRAEL JOURNAL OF MATHEMATICS 83 (1993), 253–256

# A SOLUTION TO A PROBLEM OF DUBINS AND SAVAGE

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ABSTRACT

A problem left open in Dubins and Savages' "How to Gamble if You Must" is solved.

## Introduction

In section 3, chapter 8 of Dubins and Savage [1] the question is raised as to whether or not stationary gambling houses on $(-\infty, \infty)$, with utility function $I_{\{x\geq 0\}}$, can admit value functions which are both continuous and not strictly increasing. The purpose of this paper is to provide a reasonably simple example having both properties.

## Main Section

The idea is to modify the following simple stationary gambling house which has only one gamble available at each fortune $x$:

$$\Gamma(x) = \{\gamma_x\} \quad \text{where } \gamma_x = \frac{2}{3}\delta_{x-1} + \frac{1}{3}\delta_{x+1}.$$

This gambling house has a utility function given by

$$U(x) = 1 \quad \text{if } x \geq 0; \quad = 2^{<x>} \quad \text{if } x < 0.$$

The utility function for the house should be thought of as the probability (under the optimal strategy) of attaining the positive half line, $[0, \infty)$. We will supplement this gambling house by an infinite set of gambles $\{\lambda_i\}_{i\in I}$ with the property

* Research partially supported by NSF Grant DMS 91-57461.
Received October 28, 1991 and in revised form August 19, 1992

that for each $\epsilon > 0$ there exists an $i(\epsilon) \in I$ such that $\lambda_i$ wins $\delta_i$ ($> 0$) with probability $p_i > 1 - \epsilon$. We will prove below that any stationary gambling house with these gambles available must have a continuous utility function. It should be noted that apart from being positive the $\delta_i$ and the possible losses are completely unspecified.

LEMMA 1.1: *If $\Gamma$ is a stationary gambling house with a family of gambles having the above property then its utility function must be continuous.*

*Proof:* Suppose that this is not the case. Then for some $x_0 \in R^1$

$$\lim_{x \downarrow x_0} U(x) = U(x_0+) > U(x_0-) = \lim_{x \uparrow x_0} U(x).$$

But for $\epsilon$ arbitrarily small

$$(1-\epsilon)U(x_0+) \leq (1-\epsilon)U\big(x_0 + \frac{\delta_{i(\epsilon)}}{2}\big) \leq U\big(x_0 - \frac{\delta_{i(\epsilon)}}{2}\big) \leq U(x_0-).$$

This contradiction establishes the lemma. ∎

As was observed we did not need to completely specify the winnings and the losings of the various gambles $\{\lambda_i\}$. So if we add gambles to $\Gamma$ which, while having the above property, are sufficiently unfavourable then for $x$ close to a negative integer but above it the optimum strategy should be to use the original gamble $\gamma_x$.

Let the gambling house $\Gamma^n$ be such that $\Gamma^n(x)$ contains the gambles:

$$\gamma_x = \frac{1}{3}\delta_{x+1} + \frac{2}{3}\delta_{x-1}, \quad \gamma_x^y = (1 - y^{1/4})\delta_{x+y} + y^{1/4}\delta_{x-n} \quad \text{for } y \in [0, x_n],$$

where

$$x_n = \frac{1}{[2(1-1/2^{n+1})]^4}.$$

If the gambler pursues the strategy:

(1) if at $x \geq 0$, stop;

(2) if at $x = -y$, where $0 < y < x_n$, use $\gamma_x^y$;

(3) if $x \leq -x_n$, use $\gamma_x$;

then it is clear that this strategy has utility given by the function $U^n(x)$, where

1. $U^n(x) = 1$ for $x \geq 0$,
2. $U^n(x) = \frac{1-(-x)^{1/4}}{1-(-x)^{1/4}/2^n}$ for $x \in [-x_n, 0)$,
3. $U^n(x) = 1/2$ for $x \in [-1, -x_n]$,

4. $U^n(x) = 2^{-m}U(x+m)$ for $x \in [-(m+1), -m]$.

This function is self-evidently continuous and not strictly monotonic. The rest of the paper is devoted to proving

PROPOSITION 1.1: *The utility of the gambling house $\Gamma_n$ is equal to $U_n$ for $n$ large enough.*

*Remark:* From the definition of the utility of the house, it is immediate that $U_n$ is less than or equal to the utility of $\Gamma_n$. Theorem One of Dubins and Savage [1], page 28, shows that the Proposition will be established if we can show that $U^n$ is excessive.

Before proving Proposition 1.1 we will require two lemmas.

LEMMA 1.2: *For $n$ large and all $x$, $y$, $x+y \in [-x_n, 0]$,*

$$U^n(x+y) \geq U^n(x)U^n(y).$$

*Proof:* The statement of the lemma is equivalent to the statement

$$\log(U^n(x+y)) \geq \log(U^n(x)) + \log(U^n(y)).$$

The above statement will be implied by the convexity of the function

$$u \to \log\left[\frac{1-u^{1/4}}{1-u^{1/4}/2^n}\right]$$

on the interval $[0, x_n]$. Therefore we will complete the lemma's proof by showing that this function has positive second derivative on $[0, x_n]$.

The second derivative of $\log\left[(1-u^{1/4})/(1-u^{1/4}/2^n)\right]$ with respect to $u$ is equal to

$$\frac{3}{16}u^{-7/4}\left[\frac{1}{1-u^{1/4}} - \frac{1}{2^n(1-u^{1/4}/2^n)}\right]$$
$$-\frac{1}{16}u^{-6/4}\left[\frac{1}{(1-u^{1/4})^2} - \frac{1}{2^{2n}(1-u^{1/4}/2^n)^2}\right]$$

which is greater than

$$\frac{1}{16}\frac{u^{-6/4}}{1-u^{1/4}}\left[\frac{2}{u^{1/4}} - \frac{1}{1-u^{1/4}}\right] + \frac{u^{-7/4}}{16}\left[\frac{1}{1-u^{1/4}} - \frac{3}{2^n(1-u^{1/4}/2^n)}\right].$$

The above expression is clearly strictly positive for $u \in [0, x_n]$ provided $n$ is large. ∎

LEMMA 1.3: *Let $x \in [-x_n, 0]$. For any gamble $\gamma \in \Gamma(x)$, $E_\gamma[U^n] \le U^n(x)$.*

*Proof:* We work through the gambles available in $\Gamma(x)$ systematically.

$$\gamma_x\colon E_{\gamma_x}[U^n] = \frac{1}{3} + \frac{2}{3}\frac{1}{2}U^n(x) \le \frac{2}{3}U^n(x) + \frac{2}{3}\frac{1}{2}U^n(x) = U^n(x).$$

$$\gamma_x^y,\ \text{for } y \ge -x\colon E_{\gamma_x^y}[U^n] = (1 - y^{1/4})U^n(x+y) + y^{1/4}U^n(x-n).$$

The flatness of $U^n$ to the right of integers implies that this last term is equal to

$$(1 - y^{1/4})U^n(x-x) + y^{1/4}U^n(x-n) \le (1 - (-x)^{1/4}) + (-x)^{1/4}U^n(x-n).$$

Our function $U^n$ was chosen so that this last expression is equal to $U^n(x)$.

$$\begin{aligned} &\gamma_x^y,\ \text{for } y \le -x : \\ U^n(x) - E_{\gamma_x^y}[U^n] &= U^n(x) - y^{1/4}U^n(x-n) - (1 - y^{1/4})U^n(x+y) \\ &= U^n(x) - y^{1/4}/2^n U^n(x) - (1 - y^{1/4})U^n(x+y) \\ &= (1 - y^{1/4}/2^n)\big(U^n(x) - U^n(x+y)U^n(-y)\big). \end{aligned}$$

The last term is positive by Lemma 1.2 and we are done. ∎

*Proof of Proposition 1.1:* Given the scaling properties of the function $U^n$, to establish the excessiveness of this function it suffices to show that for every $x \in [-1, 0]$ and every $\lambda \in \Gamma(x), E_\lambda[U^n] \le U^n(x)$. This has already been established for $x \in [-x_n,{}^0\,]$ in Lemma 2.2. If $x \in [-1, -x_n]$, then clearly $E_{\gamma_x}[U^n] = U^n(x) = 1/2$, while

$$E_{\gamma_x^y}[U^n] \le E_{\gamma_{-x_n}^y}[U_n] \le U_n(x_n) = U_n(x).$$

The last inequality follows from Lemma 1.3. ∎

ACKNOWLEDGEMENT: The author wishes to thank Lester Dubins for his interest and encouragement.

## References

[1] L. E. Dubins and L. J. Savage, *How to Gamble if You Must*, McGraw-Hill, New York, 1965.